Factor Polynomials

For any number of terms, check for:
- greatest common factor

For two terms, check for:
- Difference of two squares
  \[ a^2 - b^2 = (a + b)(a - b) \]
- Sum of two cubes
  \[ a^3 + b^3 = (a + b)(a^2 - ab + b^2) \]
- Difference of two cubes
  \[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]

For three terms, check for:
- Perfect square trinomials
  \[ a^2 + 2ab + b^2 = (a + b)^2 \]
  \[ a^2 - 2ab + b^2 = (a - b)^2 \]
- General trinomials
  \[ acx^2 + (ad + bc)x + bd = (ax + b)(cx + d) \]

For four or more terms, check for:
- Grouping
  \[ ax + bx + ay + by = (a + b)(x + y) \]

**Example**

Factor \(24x^2 - 42x - 45\).

First factor out the GCF to get \(24x^2 - 42x - 45 = 3(8x^2 - 14x - 15)\). To find the coefficients of the \(x\) terms, you must find two numbers whose product is \(8 \cdot (-15) = -120\) and whose sum is \(-14\). The two coefficients must be \(-20\) and \(6\). Rewrite the expression using \(-20x\) and \(6x\) and factor by grouping.

\[
8x^2 - 14x - 15 = 8x^2 - 20x + 6x - 15 \\
= 4x(2x - 5) + 3(2x - 5) \\
= (4x + 3)(2x - 5)
\]

Thus, \(24x^2 - 42x - 45 = 3(4x + 3)(2x - 5)\).

**Exercises**

Factor completely. If the polynomial is not factorable, write **prime**.

1. \(14x^2y^2 + 42xy^3\)  
   \(14xy^2(x + 3y)\)
2. \(6mn + 18m - n - 3\)  
   \((6m - 1)(n + 3)\)
3. \(2x^2 + 18x + 16\)  
   \(2(x + 8)(x + 1)\)
4. \(x^4 - 1\)  
   \((x^2 + 1)(x + 1)(x - 1)\)
5. \(35x^3y - 60x^3y\)  
   \(5x^2y(7y - 12)\)
6. \(2r^3 + 250\)  
   \(2(r + 5)(r^2 - 5r + 25)\)
7. \(100m^8 - 9\)  
   \((10m^4 - 3)(10m^4 + 3)\)
8. \(x^2 + x + 1\)  
   **prime**
9. \(c^4 + c^3 - c^2 - c\)  
   \(c(c + 1)^3(c - 1)\)
Solve Polynomial Equations If a polynomial expression can be written in quadratic form, then you can use what you know about solving quadratic equations to solve the related polynomial equation.

Example 1  Solve \( x^4 - 40x^2 + 144 = 0 \).

\[
\begin{align*}
  x^4 & - 40x^2 + 144 = 0 & \text{Original equation} \\
  (x^2)^2 & - 40(x^2) + 144 = 0 & \text{Write the expression on the left in quadratic form.} \\
  (x^2 - 4)(x^2 - 36) & = 0 & \text{Factor.} \\
  x^2 - 4 & = 0 \quad \text{or} \quad x^2 - 36 = 0 & \text{Zero Product Property} \\
  x - 2 & = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 6 = 0 \quad \text{or} \quad x + 6 = 0 & \text{Factor.} \\
  x & = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 6 \quad \text{or} \quad x = -6 & \text{Simplify.} \\
\end{align*}
\]

The solutions are \( \pm 2 \) and \( \pm 6 \).

Example 2  Solve \( 2x + \sqrt{x} - 15 = 0 \).

\[
\begin{align*}
  2x + \sqrt{x} - 15 & = 0 & \text{Original equation} \\
  2(\sqrt{x})^2 + \sqrt{x} - 15 & = 0 & \text{Write the expression on the left in quadratic form.} \\
  (2\sqrt{x} - 5)(\sqrt{x} + 3) & = 0 & \text{Factor.} \\
  2\sqrt{x} - 5 & = 0 \quad \text{or} \quad \sqrt{x} + 3 = 0 & \text{Zero Product Property} \\
  \sqrt{x} & = \frac{5}{2} \quad \text{or} \quad \sqrt{x} = -3 & \text{Simplify.} \\
\end{align*}
\]

Since the principal square root of a number cannot be negative, \( \sqrt{x} = -3 \) has no solution. The solution is \( \frac{25}{4} \) or \( 6\frac{1}{4} \).

Exercises

Solve each equation.

1. \( x^4 = 49 \)  
   \( \pm \sqrt{7}, \pm i\sqrt{7} \)

2. \( x^4 - 6x^2 = -8 \)  
   \( \pm 2, \pm \sqrt{2} \)

3. \( x^4 - 3x^2 = 54 \)  
   \( \pm 3, \pm i\sqrt{6} \)

4. \( 3t^6 - 48t^2 = 0 \)  
   \( 0, \pm 2, \pm 2i \)

5. \( m^6 - 16m^3 + 64 = 0 \)  
   \( 2, -1 \pm i\sqrt{3} \)

6. \( y^4 - 5y^2 + 4 = 0 \)  
   \( \pm 1, \pm 2 \)

7. \( x^4 - 29x^2 + 100 = 0 \)  
   \( \pm 5, \pm 2 \)

8. \( 4x^4 - 73x^2 + 144 = 0 \)  
   \( \pm 4, \pm \frac{3}{2} \)

9. \( \frac{1}{x^2} - \frac{7}{x} + 12 = 0 \)  
   \( \frac{1}{3}, \frac{1}{4} \)

10. \( x - 5\sqrt{x} + 6 = 0 \)  
    \( 4, 9 \)

11. \( x - 10\sqrt{x} + 21 = 0 \)  
    \( 9, 49 \)

12. \( x^\frac{3}{2} - 5x^\frac{1}{2} + 6 = 0 \)  
    \( 27, 8 \)